IMPROVED TARGET DETECTION IN POLARIMETRIC SAR IMAGES BY USE OF MELLIN KIND STATISTICS

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ABSTRACT

We present a modified version of the polarimetric whitening filter (PWF) for target detection in single-look complex images captured by polarimetric synthetic aperture radar (PolSAR). The modified version enables the derivation of the sampling distribution of the PWF under several product model distributions for the polarimetric scattering vector. Specifically, a constant false alarm rate (CFAR) detector is obtained under the $\mathcal{K}$-distribution and the $\mathcal{G}^0$-distribution. The PWF sampling distribution as well as efficient estimators for shape parameters characterising the radar texture are derived by use of newly proposed Mellin kind statistics for single-look complex PolSAR data.

Key words: synthetic aperture radar, polarimetry, target detection, constant false alarm rate.

1. INTRODUCTION

CFAR algorithms are obvious candidates in target detection with synthetic aperture radar (SAR) when only the background clutter can be modelled statistically, and not the target object itself. However, it is an inevitable fact that the specified false alarm rate, P_{FA}, will not match the actual P_{FA} produced by the detector. Among the contributing factors are:

- the potential mismatch between the data distribution and the chosen statistical model
- the assumption of spatially stationary statistics within the estimation window, which is in practice always violated
- the stochastic nature of the parameter estimates, which renders the detection problem a composite hypothesis test, a fact which is normally ignored

The theoretical results presented in this study improves detector performance with respect to the former and the latter of the listed issues.

The problem of model mismatch is alleviated by the provision of new and more flexible distribution models. Analytic expressions for the cumulative distribution function (cdf) of the PWF test statistic has been obtained for the $\mathcal{K}$-distributed and $\mathcal{G}^0$-distributed scattering vectors. Modeling of sea clutter was the original applications of the $\mathcal{K}$-distribution in the seminal paper by Jakeman and Pusey [1]. Later on, other product model distributions, including the $\mathcal{G}^0$-distribution, have been proposed as alternatives and validated against sea images [2].

A rigorous treatment of the composite hypothesis test problem requires derivation of the exact sampling distribution given the randomness of the parameter estimates. This is a mathematically difficult task, which is sometimes approached by finding an appropriate modification of the test statistic, such that the original cdf for the simple hypothesis test can be used. While this is a goal of future research, we settle at this point for proposing new parameter estimators with lower variance, such that the discrepancy between the specified and the actual Pfa is reduced. This has been achived by means of a new theory of Mellin kind statistics for PolSAR data on single-look complex (SLC) format [3].

The paper is organised as follows: In Section 2 we review the classical PWF, propose a modified version, and develop the theory needed to derive an asymptotic cdf for the output of the modified PWF as well as low variance estimators for the cdf parameters. Section 3 presents the derived expressions for the cdfs and the parameter estimators. Conclusions are given in Section 4.

2. THEORY

2.1. Polarimetric whitening filter

The PWF, introduced by Novak and Burl [4, 5], is one of the most important target detection algorithms for single-look complex PolSAR images. It is defined as

$$y = \mathbf{s}^H \hat{\mathbf{\Sigma}}^{-1} \mathbf{s} \tag{1}$$

where $\mathbf{s} \in \mathbb{C}^d$ is the scattering (or target) vector containing the scattering coefficients measured in d polarimetric channels, and $\mathbf{\Sigma} = \mathrm{E}\{\mathbf{s}\mathbf{s}^H\}$ is the covariance matrix of $\mathbf{s}$, which is assumed to be zero mean, but whose distribution is generally unspecified. Note that $\mathbf{\Sigma}$ must be estimated from the data, and $\hat{\mathbf{\Sigma}}$ denotes the sample mean estimator, which has traditionally been used.

The output y is a test statistic, which has been used to implement different CFAR detectors.

Gaussian case: When $\mathbf{s}$ is modelled as a complex, circular and zero mean multinormal vector, denoted $\mathbf{s} \sim \mathcal{N}_d^{\mathbb{C}}(\mathbf{0}, \mathbf{\Sigma})$, the cdf of y is a scaled F-distribution [4]:

$$y \sim \frac{nd}{n-d+1} F_{2d,2(n-d+1)} \tag{2}$$

where n is the number of samples used in $\hat{\mathbf{\Sigma}}$.

K-distributed case: When $\mathbf{s}$ is $\mathcal{K}$-distributed, written as $\mathbf{s} \sim \mathcal{K}_d^{\mathbb{C}}(\mathbf{0}, \mathbf{\Sigma}, \nu)$, the cdf of y has previously not been derived. A detector with constant yet unidentified false alarm rate has been implemented with the $N\sigma$ algorithm [5], whose detection threshold is

$$y_{thr} = \mu_y + N\sigma_y \, , \tag{3}$$

where μ_y and σ_y are the mean and standard deviation of y. The user specified parameter N is directly related to the CFAR, since it effectively places the detection threshold N standard deviations away from the mean.

2.2. Fixed-point polarimetric whitening filter

The fixed-point polarimetric whitening filter (FP-PWF) is proposed as

$$y = \mathbf{s}^H \hat{\mathbf{\Sigma}}_{\mathrm{FP}}^{-1} \mathbf{s} \, . \tag{4}$$

We modify the original PWF by applying the fixed point (FP) covariance matrix estimator proposed by Conte et al. [6] and Gini et al. [7], $\hat{\mathbf{\Sigma}}_{\mathrm{FP}}$, which is known to be both globally convergent [8], asymptotically maximum likelihood and asymptotically Wishart distributed [9]:

$$\hat{\mathbf{\Sigma}}_{FP} \overset{n\to\infty}{\sim} \mathcal{W}_d^{\mathbb{C}}\left(\left(\frac{d}{d+1}\right) n, \mathbf{\Sigma}\right) \tag{5}$$

under the product model for $\mathbf{s}$, given by

$$\mathbf{s} = \sqrt{\tau}\mathbf{x} \, . \tag{6}$$

Here τ is a positive scalar random texture variable with probability density function (pdf) $p_\tau(\tau)$, and $\mathbf{x} \sim \mathcal{N}_d^{\mathbb{C}}(\mathbf{0}, \mathbf{\Sigma})$. We thus have

$$y = \tau \cdot \left(\mathbf{x}^H \hat{\mathbf{\Sigma}}_{\mathrm{FP}}^{-1} \mathbf{x}\right) = \tau \cdot Q \, , \tag{7}$$

by introducing the quadratic form Q. The key observation is that the components of Q have distributions:

$$[\mathcal{N}_d^{\mathbb{C}}(\mathbf{0}, \mathbf{\Sigma})]^H \times \left[\frac{\mathcal{W}_d^{\mathbb{C}}(n', \mathbf{\Sigma})}{n'}\right]^{-1} \times [\mathcal{N}_d^{\mathbb{C}}(\mathbf{0}, \mathbf{\Sigma})] \, , \tag{8}$$

with $n' = dn/(d+1)$. It follows from the distribution of the complex Hotelling's T^2 statistic [10] that the asymptotic distribution of Q is

$$\begin{aligned} Q &\sim \frac{dn'}{n'-d+1} F_{2d,2(n'-d+1)} \\ &= \frac{dn}{n-d+(1/d)} F_{2d,2\left(\frac{d}{d+1}\right)(n-d+(1/d))} \, . \end{aligned} \tag{9}$$

Thus, the distribution of y is a compound F-distribution.

2.3. Mellin kind statistics

Mellin kind statistics for the PWF output can now be derived (see [3,11,12] for an introduction). The Mellin kind characteristic function (MKCF) is defined as the Mellin transform (MT) of the pdf. Since y can be written as the product in (7), the MKCF of y is decomposed as

$$\phi_y(s) = \phi_\tau(s) \cdot \phi_Q(s) \, . \tag{10}$$

This expression can be evaluated by inserting known results for the F-distribution and various texture distributions [13].

The inverse Mellin transform can be used to obtain the pdf and the cdf of the compound F-distribution. The pdf has been derived previously for gamma distributed texture [13] and is derived in the next section for inverse gamma distributed texture. Equations (7), (9) and (10) form the basis for Mellin kind statistics of single-look complex PolSAR data, first presented in [3].

The Mellin kind cumulants or log-cumulants of order r for a scalar postive random variable x were defined in [13] as

$$\kappa_r\{x\} = \left.\frac{d^r}{ds^r} \ln \phi_x(s)\right|_{s=1} \, . \tag{11}$$

The log-cumulants of y are given by [3]

$$\kappa_r\{y\} = \kappa_r\{\tau\} + \kappa_r\{Q\} \, . \tag{12}$$

For Q we have the log-cumulants [3]

$$\begin{aligned} \kappa_1\{Q\} &= \psi^{(0)}(d) - \psi^{(0)}\left(\frac{d\left(n-d+\frac{1}{d}\right)}{d+1}\right) \\ &\quad + \log\left(\frac{nd}{(d+1)} \frac{(n-d-1)}{\left(n-d+\frac{1}{d}\right)}\right) , \end{aligned} \tag{13a}$$

$$\begin{aligned} \kappa_{r>1}\{Q\} &= \psi^{(r-1)}(d) \\ &\quad - \psi^{(r-1)}\left(\frac{d\left(n-d+\frac{1}{d}\right)}{d+1}\right) . \end{aligned} \tag{13b}$$

Log-cumulants of τ for different texture distributions are found in [13].

The additivity of the log-cumulants in (12) follows from the Mellin kind statistics properties of the product model

[3,11]. Note that the terms in the sum separate the contribution of texture and speckle. It should further be noted that higher-order log-cumulants (i.e., for order $r > 1$) are independent of scale, which makes them particularly suited for estimation of the shape parameters of the texture distribution $p_\tau(\tau)$. Estimation of texture parameters can be performed by solving log-cumulant equations of one or more orders, following the method of log-cumulants, which is explained in [3]. The resulting estimators have low bias and variance compared to known methods in the literature.

3. RESULTS

In the derivations below we need a special function known as Meijer's G-function, which is defined as [14, ch. 8.2.]

$$G_{p,q}^{m,n}\left(x \,\middle|\, \begin{matrix} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{matrix}\right) = \mathcal{M}^{-1}\left\{ \frac{\prod_{i=1}^{m}\Gamma(b_i+s)}{\prod_{i=n+1}^{p}\Gamma(a_i+s)} \frac{\prod_{i=1}^{n}\Gamma(1-a_i-s)}{\prod_{i=m+1}^{q}\Gamma(1-b_i-s)} \right\}(x) \tag{14}$$

in terms of the inverse Mellin transform

$$\mathcal{M}^{-1}\{F(s)\}(x) = \frac{1}{2\pi \jmath} \int_{c-\jmath\infty}^{c+\jmath\infty} x^{-s} F(s)\, ds\,. \tag{15}$$

It is seen that Meijer's G-function is a very general function. It reduces to simpler special functions in many cases, of which many can be found in [14].

The derivation of the pdf and cdf of y when $\mathbf{s}$ has a product model distribution, such as the $\mathcal{K}$-distribution and the $\mathcal{G}^0$-distribution, is greatly simplified by use of the Mellin transform. Expressions for the MKCF of y are readily available in all these cases, since it can be evaluated from (10) by inserting known expressions for many texture distributions [13], such as

$$\phi_\tau(s) = \frac{\Gamma(\nu+s-1)}{\Gamma(\nu)} \nu^{1-s} \tag{16}$$

for a unit mean gamma distributed texture variable, $\tau \sim \gamma(\mu = 1, \nu)$, or

$$\phi_\tau(s) = \frac{\Gamma(\nu+1-s)}{\Gamma(\nu)} (\nu-1)^{s-1} \tag{17}$$

for a unit mean inverse gamma distributed texture variable, $\tau \sim \gamma^{-1}(\mu = 1, \nu)$, together with the following expression for the MKCF of Q:

$$\phi_Q(s) = \left(\frac{nd}{d+1} \frac{n-d-1}{n-d+(1/d)}\right)^{s-1} \times \frac{\Gamma(d+s-1)}{\Gamma(d)} \frac{\Gamma\left(\frac{d}{d+1}(n-d-1)+1-s\right)}{\Gamma\left(\frac{d}{d+1}(n-d+(1/d))\right)}\,. \tag{18}$$

We further need the integration property of the Mellin transform [15]:

$$f(x) \;\overset{\mathcal{M}}{\leftrightarrow}\; F(s) \tag{19}$$

$$\int_0^x f(u)\, du \;\overset{\mathcal{M}}{\leftrightarrow}\; -\frac{1}{s} F(s+1)\,. \tag{20}$$

With these relations, the MKCF $\phi_y(s)$ and the Mellin transform of the cdf, $-\phi_y(s+1)/s$, can be written as products of gamma functions and reciprocal gamma functions, like in the definition of Meijer's G-function in (14). Thus, analytic expressions for the pdf and the cdf are easily obtained in terms of Meijer's G-function.

To simplify the expressions, we provide the pdf and cdf of $z = y/c$, a linear transformation of y with the inverse constant of proportionality given by $c = dn/(n-d+(1/d))$. Pdfs and cdfs for the $\mathcal{K}$-distribution and the $\mathcal{G}^0$-distribution are given below.

K-distribution:

$$\begin{aligned} p_z(z) &= \left(\frac{N-1}{\nu d}\right) \frac{G_{1,2}^{2,1}\left(z \,\middle|\, \begin{matrix} 1-N \\ \nu, d \end{matrix}\right)}{\Gamma(\nu)\Gamma(d)\Gamma(N)} \\ &= \frac{\Gamma(\nu+N)\Gamma(d+N)}{\Gamma(\nu)\Gamma(d)\Gamma(N) \frac{\nu d}{(N-1)} \left(\frac{\nu d}{(N-1)} z\right)} e^{\frac{\nu d}{2(N-1)z}} \\ &\quad \times W_{\frac{-1-\nu-d-2N}{2}, \frac{\nu-d}{2}}\left(\frac{\nu d}{N-1} z\right) \end{aligned} \tag{21}$$

$$P_z(z) = -\left(\frac{N-1}{\nu d}\right) \frac{G_{2,3}^{3,1}\left(z \,\middle|\, \begin{matrix} 1-N, 1 \\ \nu, d, 0 \end{matrix}\right)}{\Gamma(\nu)\Gamma(d)\Gamma(N)} \tag{22}$$

where $W_{a,b}(\cdot)$ is Whittaker's W-function, $G_{p,q}^{m,n}(\cdot|\cdots)$ is Meijer's G-function, and the constant $N = d(n-d+1/d)/(d+1)$. The G-function in the pdf expression has been rewritten in terms of the Whittaker W-function because this particular inverse Mellin integral can be found in [16].

G^0-distribution:

$$p_z(z) = \frac{(\nu-1)(N-1)}{d} \frac{G_{2,1}^{1,2}\left(z \,\middle|\, \begin{matrix} 1-d, 1-N \\ \nu \end{matrix}\right)}{\Gamma(\nu)\Gamma(d)\Gamma(N)} \tag{23}$$

$$P_z(z) = -\frac{(\nu-1)(N-1)}{d} \frac{G_{2,3}^{2,2}\left(z \,\middle|\, \begin{matrix} 1-d, 1-N, 1 \\ \nu, 0 \end{matrix}\right)}{\Gamma(\nu)\Gamma(d)\Gamma(N)} \tag{24}$$

4. CONCLUSIONS

We have obtained analytic expressions for the asymptotic cdf of the FP-PWF with $\mathcal{K}$ distributed and $\mathcal{G}^0$ distributed scattering vectors. This allows us to implement

CFAR detectors using these flexible models, which reduces the problem of model mismatch to data. From the Mellin kind statistics of the modified PWF, method of log-cumulant and generalised method of log-cumulant (GMoLC) estimators for the texture parameters can be implemented. These parameters are inherited by the cdf of the FP-PWF and affect the performance of the CFAR detector. The low bias and variance of the GMoLC estimators will reduce the difference between the sampling distribution under the simple and composite hypothesis test, and therefore also the deviation of the actual false alarm rate from the specified one. Future work will include validation of the derived theory with simulated and real data.

REFERENCES

1. E. Jakeman and P. N. Pusey, "A model for non-Rayleigh sea echo," *IEEE Trans. Antennas Propag.*, vol. 24, no. 806–814, pp. 806–814, Nov. 1976.

2. Y. Delignon and W. Pieczynski, "Modeling non-Rayleigh speckle distribution in SAR images," *IEEE Trans. Geosci. Remote Sens.*, vol. 40, no. 6, pp. 1430–1435, Jun. 2002.

3. S. N. Anfinsen, "On the supremacy of logging," in *Proc. 5th Int. Workshop Sci. Appl. SAR Polarimetry and Polarimetric Interferometry (POLinSAR 2011)*, vol. ESA SP-695, Frascati, Italy, 8 pp., 24–28 Jan. 2011, in press.

4. L. M. Novak and M. C. Burl, "Optimal speckle filtering in polarimetric SAR imagery," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 26, no. 2, pp. 293–305, Mar. 1993.

5. L. M. Novak, M. C. Burl, R. D. Chaney, and G. J. Owirka, "Optimal polarimetric processing for enhanced target detection," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 29, no. 1, pp. 234–244, Jan. 1993.

6. E. Conte, A. De Maio, and G. Ricci, "Recursive estimation of the covariance matrix of a compound-Gaussian process and its application to adaptive CFAR detection," *IEEE Trans. Signal Process.*, vol. 50, no. 8, pp. 1908–1915, Aug. 2002.

7. F. Gini and M. Greco, "Covariance matrix estimation for CFAR detection in correlated heavy tailed clutter," *Signal Processing*, vol. 82, no. 12, pp. 1847–1859, Dec. 2002.

8. F. Pascal, Y. Chitour, J.-P. Ovarlez, P. Forster, and P. Larzabal, "Covariance structure maximum-likelihood estimates in compound gaussian noise: Existence and algorithm analysis," *IEEE Trans. Signal Process.*, vol. 56, no. 1, pp. 34–48, Jan. 2008.

9. F. Pascal, P. Forster, J.-P. Ovarlez, and P. Larzabal, "Performance analysis of covariance matrix estimates in impulsive noise," *IEEE Trans. Signal Process.*, vol. 56, no. 6, pp. 2206–2217, Jun. 2008.

10. N. Giri, "On the complex analogues of T^2 and R^2 tests," *Ann. Math. Statist.*, vol. 36, no. 2, pp. 664–670, Apr. 1965.

11. J.-M. Nicolas, "Introduction aux statistique de deuxième espèce: Application des logs-moments et des logs-cumulants à l'analyse des lois d'images radar," *Traitement du Signal*, vol. 19, no. 3, pp. 139–167, 2002, in French, English translation in [17].

12. S. N. Anfinsen and T. Eltoft, "Application of the matrix-variate Mellin transform to analysis of polarimetric radar images," *IEEE Trans. Geosci. Remote Sens.*, vol. 49, no. 7, Jul. 2011, 15 pp., in press.

13. J.-M. Nicolas, "Application de la transformée de Mellin: Étude des lois statistiques de l'imagerie cohérente," Ecole Nationale Supérieure des Télécommunications, Paris, France, Tech. Rep. 2006D010, 2006, in French.

14. A. P. Prudnikov, O. I. Marichev, and Y. A. Brychkov, "The Meijer G-function $G_{pq}^{mn}\left(z\middle|\begin{smallmatrix}(a_p)\\(b_q)\end{smallmatrix}\right)$," in *Integrals and Series, vol 3: More Special Functions*. Newark, U.S.: Gordon and Breach, 1990.

15. J. Bertrand, P. Bertrand, and J.-P. Ovarlez, "The Mellin transform," in *The Transform and Applications Handbook*, 2nd ed., A. D. Poularikas, Ed. Boca Raton, USA: CRC Press, 2000, ch. 11.

16. F. Oberhettinger, *Tables of Mellin Transforms*. New York, U.S.: Springer-Verlag, 1974.

17. S. N. Anfinsen, "Statistical analysis of multilook polarimetric radar images with the Mellin transform," Ph.D. dissertation, University of Tromsø, Tromsø, Norway, May 2010.