# Analysis of Multilook Polarimetric Radar Data with the Matrix-Variate Mellin Transform

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## Abstract

In this paper we propose to use a matrix-variate Mellin transform in the statistical analysis of multilook polarimetric radar data. The domain of the transform integral is the cone of complex positive definite matrices, which allows for transformation of the sample covariance matrix distributions used to model multilook polarimetric radar data. Based on the matrix-variate Mellin transform, an alternative characteristic function is defined, from which we can retrieve a new kind of matrix log-moments and log-cumulants. We show that the matrix log-cumulants are of great value to the analysis of polarimetric radar data, and can be used to derive low bias and variance estimators for the distribution parameters.

## 1 Introduction

Polarimetric radar is an important remote sensing instrument which is able to discriminate between different scattering mechanisms and characterise physical properties of the target that cannot be determined from single polarisation measurements. To fully utilise the polarimetric information captured, the complex correlations between all polarimetric channels must be analysed, including all intensity and phase information. This requires relatively complicated data models, that together with the speckle phenomenon, inherent to all types of coherent imaging, make analysis of polarimetric radar data a challenging task.

The measurements of imaging radar are efficiently modelled by the doubly stochastic product model [1]. Nicolas made a considerable contribution to the applicability of this model, when he demonstrated the facility of the Mellin transform (MT) to analysis of the complicated radar data distribution models in single polarisation case [2, 3]. He introduced a new theoretical framework by replacing the Fourier transform with the MT in the definition of the characteristic function (CF) and cumulant generating function (CGF) of radar signal random variables. This framework was originally coined second kind statistics, but we shall refer to it as Mellin kind statistics (MKS). From the resulting Mellin kind CF and CGF one can retrieve the statistics known as log-moments and log-cumulants.

The most important development under this framework is the method of log-cumulants (MoLC) for parameter estimation, which Nicolas applied to a number of doubly stochastic distributions, as well as the positive alpha-stable distribution, and members of the generalised gamma distribution family (e.g., the Weibull and log-normal distribution) [2, 3]. The list of recently proposed synthetic aperture radar image analysis algorithms employing the MoLC covers diverse applications such as statistical modelling, speckle filtering, classification, segmentation, change detection, interferometric coherence estimation, and image compression (See [4] for references).

We here extend the theory of MKS to the matrix-variate case which describes multilook polarimetric radar data. This is done by introducing a matrix-variate version of the MT [5], which is used to define a Mellin kind CF and CGF of random matrices. We then show how matrix log-moments and matrix log-cumulants can be obtained from the Mellin kind matrix CF and CGF, respectively.

In Section 2 we describe the data delivered by polarimetric radars, together with the probability density functions (PDFs) commonly used to model the data. In Section 3 we present the proposed framework of analysis based on matrix-variate Mellin transform. Applications of the theory are shown in Section 4, before conlusions are given in Section 5. We denote scalar values as lower or upper case standard weight characters, vectors are lower case boldface characters, and matrices are upper case boldface characters. The distinction between random variables and instances of random variables must in most cases be ascertained through context.

## 2 Polarimetric Radar Data

The measurable of a polarimetric radar is the Sinclair scattering matrix $\mathbf{S}$, or equivalently, the scattering vector $\mathbf{s}$, which is simply the vectorised version. We have

$$\mathbf{S} = \begin{bmatrix} S_{xx} & S_{xy} \\ S_{yx} & S_{yy} \end{bmatrix}, \quad \mathbf{s} = \begin{bmatrix} S_{xx} \\ S_{xy} \\ S_{yx} \\ S_{yy} \end{bmatrix} = \mathrm{vec}(\mathbf{S}^T) \qquad (1)$$

where $(\cdot)^T$ and $\mathrm{vec}(\cdot)$ denote the transposition and vectorisation operator. The entries of $\mathbf{S}$ and $\mathbf{s}$ are the scattering coefficients of the $d$ polarimetric channels. These are complex-valued, dimensionless numbers that describe the transformation of incident to backscattered electromagnetic field for all combinations of the two orthogonal transmit and receive polarisations, denoted by $x$ and $y$.

In the following, we shall only be concerned with multilook complex data. Multilooking is an averaging process, applied either during or after focusing of the radar image, which reduces the data amount and suppresses the noise-like effect of the interference, known as speckle, at the expense of reduced spatial resolution. Thus, polarimetric radar data is represented in the intensity domain by:

$$\mathbf{C} = \frac{1}{L}\sum_{\ell=1}^{L} \mathbf{s}_\ell \mathbf{s}_\ell^H \tag{2}$$

or a linearly transformed version of $\mathbf{C}$, where $L$ is the equivalent number of looks averaged in the process and $(\cdot)^H$ is the Hermitian (conjugate transposition) operator. We refer to $\mathbf{C} \in \Omega_+ \subset \mathbb{C}^{d\times d}$ as the multilook polarimetric covariance matrix, and note that $\mathbf{C}$ is a random matrix defined on the cone $\Omega_+$ of positive definite complex Hermitian matrices.

We base our work upon the multilook polarimetric product model [6], which decomposes $\mathbf{C}$ as

$$\mathbf{C} = T\widetilde{\mathbf{W}} = T(\mathbf{W}/L)\,. \tag{3}$$

The strictly positive and unit mean scalar random variable $T$ models texture, which is here defined as spatial variation in the mean backscatter due to target variability. The other component is the random matrix $\widetilde{\mathbf{W}} \sim \mathbf{W}/L$, where $\mathbf{W} \sim \mathcal{W}_d^{\mathbb{C}}(L, \mathbf{\Sigma})$ follows a complex Wishart distribution with scale matrix $\mathbf{\Sigma} = \mathrm{E}\{\mathbf{W}\}/L$ and $L$ degrees of freedom [6], and the scaled Wishart matrix $\widetilde{\mathbf{W}}$ models the variability of $\mathbf{C}$ attributed to fully developed speckle.

The simplest model for the PDF of $\mathbf{C}$ assumes that the scattering coefficients are jointly circular complex Gaussian. This is strictly justified only for homogeneous regions of the image characterised by fully developed speckle and no texture (i.e., $T = 1$ is constant), and leads to the scaled complex Wishart distribution:

$$p_{\mathbf{C}}(\mathbf{C}; L, \mathbf{\Sigma}) = \frac{L^{Ld}}{\Gamma_d(L)} \frac{|\mathbf{C}|^{L-d}}{|\mathbf{\Sigma}|^L} \operatorname{etr}(-L\mathbf{\Sigma}^{-1}\mathbf{C}) \tag{4}$$

where $|\cdot|$ is the determinant, $\operatorname{etr}(\cdot) = \exp(\operatorname{tr}(\cdot))$, $\operatorname{tr}(\cdot)$ is the trace operator, $\Gamma_d(L)$ is the multivariate gamma function of the complex kind [6], and $L \geq d$ assures that $\mathbf{C}$ is nonsingular. When the PDF of $T$ is not degenerate, we obtain more complicated distributions for $\mathbf{C}$, such as the matrix-variate $\mathcal{K}$ distribution [6] (for a $\Gamma$ distributed $T$), $\mathcal{G}^0$ distribution [6] (inverse $\Gamma$ distributed $T$), $\mathcal{U}$ distribution [7] ($\mathcal{F}$ distributed $T$), $\mathcal{W}$ distribution [4] ($\beta$ distributed $T$) and $\mathcal{M}$ distribution (inverse $\beta$ distributed $T$). These are derived from

$$p_{\mathbf{C}}(\mathbf{C}) = \int_0^\infty p_{\mathbf{C}|T}(\mathbf{C}|t)p_T(t)\,dt \tag{5}$$

where $\mathbf{C}|T \sim s\mathcal{W}_d^{\mathbb{C}}(L, \mathbf{\Sigma})$ follows the scaled Wishart distribution from (4).

# 3 Theory

## 3.1 Matrix-Variate Mellin Kind Statistics

Mathai defined in [5] a generalised matrix transform (the so-called M-transform) for matrix-valued functions, which he has also referred to as a generalised MT. In the following, let $f(\mathbf{Z})$ be a real-valued scalar function defined on $\Omega_+$, and let $f$ be symmetric in the sense $f(\mathbf{ZV}) = f(\mathbf{VZ})$ where $\mathbf{V}, \mathbf{Z} \in \Omega_+$. Whenever the integral exists, the complex matrix-variate MT of $f(\mathbf{Z})$ is defined as

$$\mathcal{M}\{f(\mathbf{Z})\}(s) = \int_{\Omega_+} |\mathbf{Z}|^{s-d} f(\mathbf{Z})\,d\mathbf{Z}\,. \tag{6}$$

We note that $\mathcal{M}\{f(\mathbf{Z})\}(s)$ is a function of a complex scalar transform variable $s$, whereas $f(\mathbf{Z})$ is defined on a matrix space, thus the transform is not unique and has no inverse. The symmetry requirement restricts the functions (6) can be applied to, but does not pose any problem for the product model PDFs. We may therefore use the transform to define MKS for the complex matrix-variate case.

The Mellin kind CF of a random matrix $\mathbf{Z}$ is defined as

$$\phi_{\mathbf{Z}}(s) = \mathrm{E}\{|\mathbf{Z}|^{s-d}\} = \mathcal{M}\{p_{\mathbf{Z}}(\mathbf{Z})\}(s) \tag{7}$$

when $\mathbf{Z}$ and its PDF, $p_{\mathbf{Z}}(\mathbf{Z})$, satisfy all requirements of (6). The $\nu$th-order Mellin kind moment of $\mathbf{Z}$ is

$$\mu_\nu\{\mathbf{Z}\} = \left.\frac{d^\nu}{ds^\nu}\phi_{\mathbf{Z}}(s)\right|_{s=d}. \tag{8}$$

If all Mellin kind matrix moments exist, the Mellin kind CF can be written as the power series expansion

$$\begin{aligned}\phi_{\mathbf{Z}}(s) &= \int_{\Omega_+} e^{(s-d)\ln|\mathbf{Z}|} p_{\mathbf{Z}}(\mathbf{Z})\,d\mathbf{Z} \\ &= \sum_{\nu=0}^{\infty} \frac{(s-d)^\nu}{\nu!}\mu_\nu\{\mathbf{Z}\}\end{aligned} \tag{9}$$

in terms of the $\mu_\nu\{\mathbf{Z}\}$. The derivation of (9) reveals that

$$\mu_\nu\{\mathbf{Z}\} = \mathrm{E}\{(\ln|\mathbf{Z}|)^\nu\} = \int_{\Omega_+} (\ln|\mathbf{Z}|)^\nu p_{\mathbf{Z}}(\mathbf{Z})\,d\mathbf{Z} \tag{10}$$

which justifies the denotation of $\mu_\nu\{\mathbf{Z}\}$ as a matrix log-moment (MLM).

The Mellin kind CGF of $\mathbf{Z}$ is defined as

$$\varphi_{\mathbf{Z}}(s) = \ln\phi_{\mathbf{Z}}(s)\,. \tag{11}$$

The $\nu$th-order Mellin kind cumulant of $\mathbf{Z}$ is

$$\kappa_\nu\{\mathbf{Z}\} = \left.\frac{d^\nu}{ds^\nu}\varphi_{\mathbf{Z}}(s)\right|_{s=d}. \tag{12}$$

When all Mellin kind matrix moments exist, the Mellin kind CGF can be expanded as

$$\varphi_{\mathbf{Z}}(s) = \ln\phi_{\mathbf{Z}}(s) = \sum_{\nu=0}^{\infty}\frac{(s-d)^\nu}{\nu!}\kappa_\nu\{\mathbf{Z}\} \tag{13}$$

in terms of the $\kappa_\nu\{\mathbf{Z}\}$, that are also called matrix log-cumulants (MLCs).
To prove a fundamental property of the matrix-variate MT, we need to introduce the matrix-variate Mellin convolution. Let $f(\mathbf{U})$ and $g(\mathbf{U})$ be two functions defined on $\Omega_+$. Further assume that $\mathbf{U},\mathbf{V}\in\Omega_+$ and $f(\mathbf{U}\mathbf{V})=f(\mathbf{V}\mathbf{U})$. We define the Mellin convolution of $f$ and $g$ as

$$(f\,\hat{\star}\,g)(\mathbf{U})=\int_{\Omega_+}|\mathbf{V}|^{-d}f(\mathbf{V}^{-\frac{1}{2}}\mathbf{U}\mathbf{V}^{-\frac{1}{2}})g(\mathbf{V})\,d\mathbf{V}\,. \quad (14)$$

Under the same assumptions we have proven [4] that

$$\mathcal{M}\{(f\,\hat{\star}\,g)(\mathbf{U})\}(s)=\mathcal{M}\{f(\mathbf{U})\}(s)\cdot\mathcal{M}\{g(\mathbf{U})\}(s)\,. \quad (15)$$

Thus, there is a complete analogy with the MKS derived in the univariate case, and all the attractive features outlined in [2] are inherited. A matrix-variate correlation and cross-correlation has also been defined [4].

### 3.2 Application to the Product Model

Recall the polarimetric product model as $\mathbf{C}=T\widetilde{\mathbf{W}}$. By writing $\mathbf{C}=\mathbf{T}\widetilde{\mathbf{W}}$, where $\mathbf{T}=T\mathbf{I}_d$ with $\mathbf{I}_d$ the $d\times d$ identity matrix, we express $\mathbf{C}$ as a product of two random matrices. We easily prove that the PDF of $\mathbf{C}$ is $p_{\mathbf{C}}(\mathbf{C})=(p_{\mathbf{T}}\,\hat{\star}\,p_{\widetilde{\mathbf{W}}})(\mathbf{C})$, and it follows from (15) that

$$\phi_{\mathbf{C}}(s)=\phi_{\mathbf{T}}(s)\cdot\phi_{\widetilde{\mathbf{W}}}(s) \quad (16)$$
$$\varphi_{\mathbf{C}}(s)=\varphi_{\mathbf{T}}(s)+\varphi_{\widetilde{\mathbf{W}}}(s) \quad (17)$$
$$\kappa_\nu\{\mathbf{C}\}=\kappa_\nu\{\mathbf{T}\}+\kappa_\nu\{\widetilde{\mathbf{W}}\}\,. \quad (18)$$

The obvious significance is that MKS separates the contributions of texture and speckle. Additionally, mathematically simple expressions are obtained for (16)–(18) under distributions that are relatively complicated.
The Mellin kind CF of $\mathbf{W}\sim\mathcal{W}_d^{\mathbb{C}}(L,\boldsymbol{\Sigma})$ is [4]

$$\phi_{\mathbf{W}}(s)=\mathcal{M}\{p_{\mathbf{W}}(\mathbf{W})\}(s)=\frac{\Gamma_d(L+s-d)}{\Gamma_d(L)}|\boldsymbol{\Sigma}|^{s-d} \quad (19)$$

and we can show from (12) that the MLCs of $\widetilde{\mathbf{W}}$ become

$$\kappa_1\{\tilde{\mathbf{W}}\}=\psi_d^{(0)}(L)+\ln|\boldsymbol{\Sigma}|-d\ln L \quad (20)$$
$$\kappa_\nu\{\tilde{\mathbf{W}}\}=\psi_d^{(\nu-1)}(L)\,;\quad \nu>1 \quad (21)$$

where we have introduced the $\nu$th-order multivariate polygamma function as

$$\psi_d^{(\nu)}(L)=\sum_{i=0}^{d-1}\psi^{(\nu)}(L-i) \quad (22)$$

with the ordinary polygamma functions given by $\psi^{(\nu-1)}(L)=d^\nu\ln\Gamma(L)/dL^\nu$.
For the texture part, we find that

$$\phi_{\mathbf{T}}(s)=\sum_{\nu=0}^{\infty}\frac{[d(s-d)]^\nu}{\nu!}\mu_\nu\{T\} \quad (23)$$

which implies that

$$\kappa_\nu\{\mathbf{T}\}=d^\nu\kappa_\nu\{T\}\,. \quad (24)$$

Here $\mu_\nu\{T\}$ and $\kappa_\nu\{T\}$ are the univariate log-moments and log-cumulants of $T$. These have been derived in [3] for various texture distributions. The derivation of (24) relies on the general relation between moments and cumulants:

$$\kappa_\nu\{\cdot\}=\mu_\nu\{\cdot\}-\sum_{i=1}^{\nu-1}\binom{\nu-1}{i-1}\kappa_i\{\cdot\}\mu_{\nu-i}\{\cdot\}\,. \quad (25)$$

We then arrive at the MLCs of $\mathbf{C}$:

$$\kappa_1\{\mathbf{C}\}=\psi_d^{(0)}(L)+\ln|\boldsymbol{\Sigma}|-d(\ln L-\kappa_1\{T\}) \quad (26)$$
$$\kappa_\nu\{\mathbf{C}\}=\psi_d^{(\nu-1)}(L)+d^\nu\kappa_\nu\{T\}\,;\quad \nu>1 \quad (27)$$

derived under the product model for a texture variable with unspecified distribution.
Finally note that sample MLMs, denoted $\langle\mu_\nu\{\mathbf{C}\}\rangle$, are calculated with the sample mean estimator

$$\langle\mu_\nu\{\mathbf{C}\}\rangle=\frac{1}{n}\sum_{i=1}^{n}(\ln|\mathbf{C}_i|)^\nu \quad (28)$$

given a sample of $n$ covariance matrices. We use (25) to convert these into sample MLCs: $\langle\kappa_\nu\{\mathbf{C}\}\rangle$.

## 4 Applications

To promote intuition about applications of the MKS framework, we use the same kind of log-cumulant diagrams as in the univariate case [2, 3]. The MLC diagram shows: (i) the manifolds spanned by the theoretical MLCs that can be attained under given distribution models, and (ii) points that represent the empirical sample MLCs computed from data samples. The dimension of a given manifold equals the number of free parameters in the texture distribution $p_T(t)$ associated with that product model distribution. Thus, the manifolds are zero-dimensional for the Wishart distribution (i.e., a point), one-dimensional for the $\mathcal{K}$ and $\mathcal{G}^0$ distribution (curves), and two-dimensional for the $\mathcal{U}$, $\mathcal{W}$ and $\mathcal{M}$ distribution (surfaces).
This is illustrated in **Figure 1**, which displays $\kappa_3\{\mathbf{C}\}$ against $\kappa_2\{\mathbf{C}\}$. We have shown in (27) that under the polarimetric product model, MLCs of order higher than two are independent of the scale matrix $\boldsymbol{\Sigma}$. Assuming that $L$ is a global constant for the data set, this diagram shows the solitary impact of the texture parameters upon the models. Thus, it provides insight about how the texture parameters can be estimated.
Given a sample of covariance matrices, we can compute sample MLCs and plot them as points in MLC space. This has been done in Figure 1 for samples of forest (green points), ocean (blue), urban area (red) and cropland (black), taken from different polarimetric NASA/JPL AIRSAR C-band images. Multiple points were obtained for each area by bootstrap sampling of the $\mathbf{C}$ samples.

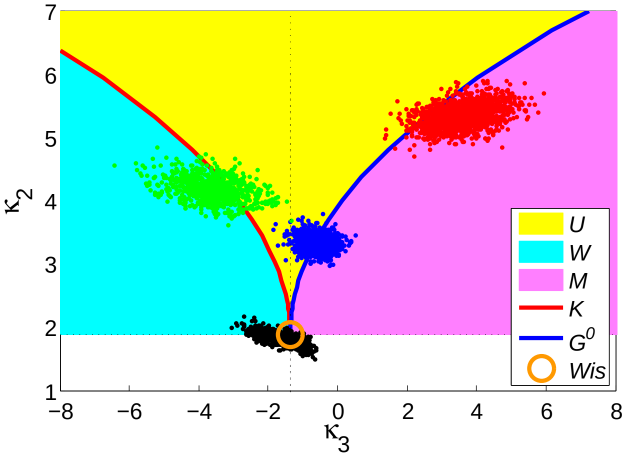


**Figure 1:** Matrix log-cumulant diagram

Parameter estimation can be visualised as a projection of the sample MLCs onto the manifolds representing the models. The manifolds are functions of the texture parameters, and the parameter value at the projection point is assigned as an estimate. The method of matrix log-cumulants (MoMLC) requires as many MLC equations as we have unknown parameters. We then insert sample MLCs instead of theoretical MLCs and solve for the parameters.

We have also proposed another estimation method that utilises more cumulants than the MoMLC in order to capture more information. Let $\boldsymbol{\kappa}$ be a vector of different MLCs with selected orders and $\langle\boldsymbol{\kappa}\rangle$ the corresponding vector of sample MLCs, such that $\mathrm{E}\{\langle\boldsymbol{\kappa}\rangle\} = \boldsymbol{\kappa}$. We have further derived an asymptotic (large sample) approximation for the covariance matrix $\mathbf{K} = \mathrm{E}\{(\langle\boldsymbol{\kappa}\rangle - \boldsymbol{\kappa})(\langle\boldsymbol{\kappa}\rangle - \boldsymbol{\kappa})^T\}$ [4]. Knowing that $\boldsymbol{\kappa}$ depends on the texture parameters through $T$, we define the maximum asymptotic likelihood (MAL) estimator, whose name is explained in [4]:

$$\hat{\boldsymbol{\theta}} = \min_{\boldsymbol{\theta}}\{Q\} = \min_{\boldsymbol{\theta}} \left\{ n\big(\langle\boldsymbol{\kappa}\rangle - \boldsymbol{\kappa}\big)^T \mathbf{K}^{-1} \big(\langle\boldsymbol{\kappa}\rangle - \boldsymbol{\kappa}\big) \right\} \quad (29)$$

where $\boldsymbol{\theta}$ is a vector of the parameters to be estimated.

**Figure 2** shows bias (full lines) and variance (dashed lines) as function of sample size $n$ for three estimators of the matrix-variate $K$ distribution texture parameter. The MAL estimator is superior, followed by the MoMLC estimator, and then the sequential method of log-cumulant (SMoLC) estimator as the inferior. The latter is an average of the univariate method of log-cumulant estimates [2, 3] obtained from each polarimetric channel.

We finally mention that the test statistic $Q$ can be used to measure goodness-of-fit of the product models (See [4] for details). The procedure can be viewed as measuring the distance between sample MLCs and the model manifolds.

# 5 Conclusions

We have extended the theory of Mellin kind statistics to the matrix-variate case describing polarimetric radar data. The Mellin transform of the polarimetric covariance matrix can be interpreted as a Laplace transform of the matrix determinant applied on logarithmic scale, which explains why it resolves the product model and separates the texture and speckle contribution. A less intuitive result is that the resulting Mellin kind characteristic functions, moments and cumulants have a mathematically simple form, and provide parameter estimates with superb statistical properties. This promotes the matrix-variate Mellin transform as a natural tool for analysis of multilook polarimetric radar data.

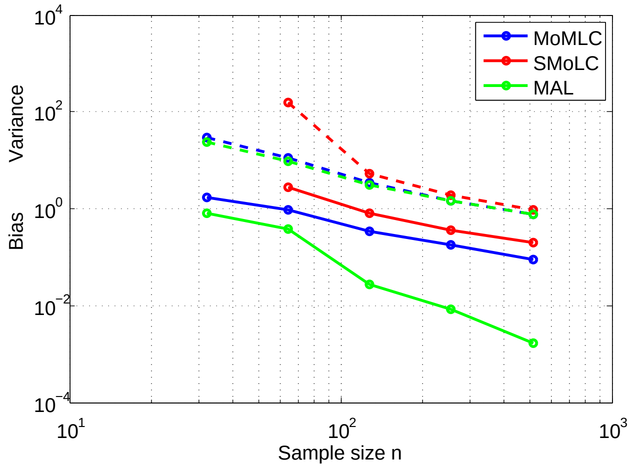


**Figure 2:** Statistical properties of MKS-based estimators

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