ON THE SUPREMACY OF LOGGING

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ABSTRACT

A theoretical framework for the statistical analysis of amplitude and intensity images acquired with synthetic aperture radar has previously been introduced by Nicolas. The framework is founded on use of the Mellin transform, and will therefore be referred to as Mellin kind statistics. The approach was recently extended to multilook polarimetric radar data, and is here further expanded to include asymptotical statistics for single-look complex polarimetric data. A common feature of the approaches is that they rely on some kind of logarithmic transformation of the data, which proves to be a key factor in statistical analysis of radar images. The application of Mellin kind statistics to estimation of shape parameters in probability density functions for the scattering vector is demonstrated with simulations.

Key words: synthetic aperture radar; polarimetry; statistical modelling; Mellin transform; parameter estimation.

1. INTRODUCTION

In a seminal paper [1], Nicolas provided a groundbreaking theory for the statistical analysis of the probability density functions (PDFs) used to model single channel amplitude and intensity images produced by synthetic aperture radar (SAR). This theory, referred to in this paper as Mellin kind statistics (MKS), is founded on the definition of the Mellin kind characteristic function as the Mellin transform of the pdf, whereas the traditional characteristic function is defined in terms of the Laplace or Fourier transform.

From the Mellin kind characteristic function one may retrieve Mellin kind moments and cumulants, that are moments and cumulants of the amplitude or intensity computed on logarithmic scale. The use of the Mellin transform proves equally natural in the analysis of the product model commonly used to model radar data, as the Laplace or Fourier transform are in the analysis of the additive model, which is more familiar in the general statistical signal processing literature.

In this paper it is snown how the logarithmic transformation, through its intrinsic connection with the Mellin transform, leads to mathematically elegant and tractable expressions as well as efficient tools in parameter estimation for the distributions of radar data. Section 2 reviews how Nicolas' univariate MKS framework has recently been extended to cover multilook polarimetric radar data on covariance matrix format and the estimation procedures provided [2, 3]. Section 3 presents an asymptotic extension of the MKS framework to polarimetric single-look complex (SLC) data and a discussion of estimators derived from the new theory. Estimation results with simulated data are shown in Section 4 before conclusions are given in Section 5.

2. REVIEW

This section starts with a review of the mathematical tools needed in the following derivations, namely the Mellin transform and its matrix-variate extension. Then follows a description of the different product models applied in the analysis of single polarisation amplitude or intensity data, SLC polarimetric data, and multilook complex polarimetric data. This all leads up to the definitions of MKS for the data formats described by the mentioned models. These results all illustrates the natural place of the logarithmic transformation in statistical analysis of radar image data. They are also partly needed in the extensions of the theory presented in the next section.

2.1. Mellin transform

The Mellin transform of the real-valued function $f(x)$ defined on $\mathbb{R}^+$ is

$$F(s) = \mathcal{M}\{f(x)\}(s) = \int_0^\infty x^{s-1} f(x)\, dx \qquad (1)$$

where $s \in \mathbb{C}$ is a complex transform variable. Under certain restrictions on $f(x)$, $F(s)$ will be analytic in a strip parallel to the imaginary axis. It is worth pointing out that the Mellin transform is commonly interpreted as a Laplace transform computed on logarithmic scale. This is seen by rewriting (1) as

$$F(s) = \int_0^\infty e^{s \log x} f(x) \frac{dx}{x} = \int_0^\infty e^{sy} f(e^y)\, dy \qquad (2)$$

using the substitution $y = \log(x)$.

A complex matrix-variate Mellin transform was introduced by Mathai in [4]. Let $f(\mathbf{X})$ be a real-valued scalar function defined on a cone Ω_+ of complex, positive definite and Hermitian matrices with dimension $d \times d$, and let the function f be symmetric in the sense that $f(\mathbf{XY}) = f(\mathbf{YX})$, where $\mathbf{X}, \mathbf{Y} \in \Omega_+$. The Mellin transform of $f(\mathbf{X})$ is then given by

$$F(s) = \mathcal{M}\{f(\mathbf{X})\}(s) = \int_{\Omega_+} |\mathbf{X}|^{s-d} f(\mathbf{X})\, d\mathbf{X} \tag{3}$$

which can again be related to the Laplace transform by

$$F(s) = \int_{\Omega_+} e^{s \log|\mathbf{X}|} |\mathbf{X}|^{-d}\, d\mathbf{X} = \int_0^\infty e^{sy}\, dy \tag{4}$$

noting that $|\mathbf{X}|^{-d}$ is the Jacobian determinant of the transformation $y = \log|\mathbf{X}|$. This reflects an important difference between the univariate and the matrix-variate transform; The former is a bijective and is associated with an inverse transform, while the latter is a surjective (onto), and cannot be inverted.

2.2. Product model

The doubly stochastic product model was introced in the radar context by Jakeman and Pusey (cite); Let Y be a random variable which could represent a single channel multilook amplitude or multilook intensity. It can be decomposed as

$$Y = \tau X \tag{5}$$

where τ is a positive random variable representing texture, with an unspecified pdf $p_\tau(\tau)$. The random variable X represents fully developed speckle and will thus have a Nakagami distribution in the amplitude case and a gamma distribution in the intensity case.

In the SLC case, the scattering vector $\mathbf{s}$ can be modelled as

$$\mathbf{s} = \sqrt{\tau}\mathbf{x} \tag{6}$$

where τ is again a texture variable, while speckle is now modelled by the complex, circular and multinormal vector $\mathbf{x} \sim \mathcal{N}_d^{\mathbb{C}}(\mathbf{0}, \mathbf{\Sigma})$ with zero mean and covariance matrix $\mathbf{\Sigma} = E\{\mathbf{s}\mathbf{s}^H\}$. This is equivalent to the spherically invariant random vector (SIRV) model, a term used by several authors [5, 6].

Finally, the multilook complex (MLC) product model decomposes the polarimetric covariance matrix $\mathbf{C}$ (or equivalently the coherency matrix) as

$$\mathbf{C} = \tau \mathbf{X} \tag{7}$$

where speckle is represented by the random matrix $\mathbf{X}$, such that $L\mathbf{X} \sim \mathcal{W}_d^{\mathbb{C}}(L, \mathbf{\Sigma})$ follows the complex Wishart distribution with L equivalent number of looks. The SLC product model can be translated into the MLC product model by multilooking (6). However, the equivalence between (6) and (7) relies on the texture being constant within each multilook cell. Under this assumption, the shape parameter(s) of the texture distribution will still be different, although related.

The pdfs of the product model outputs depend on the pdf of τ and is derived through a multiplicative convolution, also known as a Mellin convolution. Some important examples of product model distributions are the $\mathcal{K}$-distribution, the $\mathcal{G}^0$-distribution and the $\mathcal{U}$-distribution, obtained when τ is modelled with the gamma, inverse gamma and F-distribution, respectively. Distinct, although related, distributions are obviously obtained for the different data formats, indicating that the mentioned distributions are actually distribution families classified according to the texture distribution used to derive them.

2.3. Mellin kind statistics: univariate case

Because of the domain of the Mellin transform integral, it can be directly applied to pdfs for amplitude and intensity, since these are defined on $\mathbb{R}^+$. This was exploited in [1], where Nicolas introduced the Mellin kind characteristic function[1] as

$$\phi_Y(s) = E\{Y^{s-1}\} = \mathcal{M}\{p_Y(y)\}(s) \tag{8}$$

by replacing the Laplace (or Fourier) transform in the definition of classical characteristic function with the Mellin transform. The Maclaurin series expansion of the exponential function is used to show

$$\begin{aligned} \phi_Y(s) &= \int_0^\infty e^{(s-1)\log y} p(y)\, dy \\ &= \sum_{r=0}^\infty \frac{(s-1)^r}{r!} \int_0^\infty (\log y)^r p_Y(y)\, dy \\ &= \sum_{r=0}^\infty \frac{(s-1)^r}{r!} \mu_r\{Y\} \end{aligned} \tag{9}$$

which proves that $\phi_Y(s)$ can be expanded in terms of log-moments, defined as $\mu_r\{Y\} = E\{(\log Y)^r\}$, provided they all exist. From (9), it is also seen that the log-moments can be retrieved from $\phi_Y(s)$ by

$$\mu_r\{Y\} = \left.\frac{d^r}{ds^r}\phi_Y(s)\right|_{s=1}. \tag{10}$$

The Mellin kind cumulant generating function is defined as $\varphi_Y(s) = \log \phi_Y(s)$. This function can be expanded as

$$\varphi_Y(s) = \sum_{r=0}^\infty \frac{(s-1)^r}{r!} \kappa_r\{Y\} \tag{11}$$

[1] The original terms used by Nicolas were *second kind statistics*, *second kind characteristic function*, and so on.

with the coefficients $\kappa_r\{Y\}$ referred to as log-cumulants, provided all log-cumulants exist. This is equivalent to requiring that all log-moments exist, since $\kappa_r\{Y\}$ is a polynomial in the log-moments up to the same order. The log-cumulants are retrived from (11) by

$$\kappa_r\{Y\} = \left.\frac{d^r}{ds^r}\varphi_Y(s)\right|_{s=1} . \tag{12}$$

The first three relations between log-moments and log-cumulants are

$$\kappa_1 = \mu_1 \tag{13}$$
$$\kappa_2 = \mu_2 - \mu_1^2 \tag{14}$$
$$\kappa_3 = \mu_3 - 3\mu_1\mu_2 + 2\mu_1^3 \tag{15}$$

These relations are valid for moments and cumulants in general. The reference to the random variate was dropped for this reason.

2.4. Mellin kind statistics: matrix-variate case

The domain of the complex matrix-variate Mellin transform in (3) coincides with the domain of the covariance matrix distributions. It can thus be used to define a Mellin kind characteristic function for the complex matrix-variate case:

$$\phi_{\mathbf{C}}(s) = E\{|\mathbf{C}|^{s-d}\} = \mathcal{M}\{p_{\mathbf{C}}(\mathbf{C})\}(s) \tag{16}$$

which can be expanded into

$$\phi_{\mathbf{C}}(s) = \sum_{r=0}^{\infty} \frac{(s-d)^r}{r!}\mu_r\{\mathbf{C}\} \tag{17}$$

in terms of the matrix log-moments $\mu_r\{\mathbf{C}\} = E\{\log|C|\}$, provided they exist. The matrix log-moments are retrieved from

$$\mu_r\{\mathbf{C}\} = \left.\frac{d^r}{ds^r}\phi_{\mathbf{C}}(s)\right|_{s=d} . \tag{18}$$

Again, we see the Mellin transform implies that the data are analysed on logarithmic scale, in this case with the determinant condensing the matrix information into a scalar.

The Mellin kind cumulant generating function in the complex matrix-variate case becomes $\varphi_{\mathbf{C}}(s) = \log\phi_{\mathbf{C}}(s)$, whose expansion is

$$\varphi_{\mathbf{C}}(s) = \sum_{r=0}^{\infty} \frac{(s-d)^r}{r!}\kappa_r\{\mathbf{C}\} \tag{19}$$

in terms of the matrix log-cumulants $\kappa_r\{\mathbf{C}\}$, provided they exist. The matrix log-cumulants are retrieved from

$$\kappa_r\{\mathbf{C}\} = \left.\frac{d^r}{ds^r}\varphi_{\mathbf{C}}(s)\right|_{s=d} . \tag{20}$$

2.5. Application to the product model

The Fourier transform plays an important role in signal processing with the additive signal model due to its convolution property: A convolution in the input domain corresponds to a multiplication in the Fourier domain. For the product model, the Mellin transform has the same properties, as we shall see.

It was shown in [1] that for the univariate product model, we have the following relations:

$$p_Y(y) = p_\tau(\tau) \hat{\star} p_X(x) \tag{21a}$$
$$\phi_Y(s) = \phi_\tau(s) \cdot \phi_X(s) \tag{21b}$$
$$\varphi_Y(s) = \varphi_\tau(s) + \varphi_X(s) \tag{21c}$$
$$\kappa_r\{Y\} = \kappa_r\{\tau\} + \kappa_r\{X\} . \tag{21d}$$

The pdf of Y is obtained from the pdfs of τ and X through a Mellin convolution, denoted $\hat{\star}$. The Mellin kind characteristic function of Y decomposes as a product, the Mellin kind cumulant generating function as a sum, and the log-cumulants also as a sum.

Analogous properties was proven for the complex matrix-variate case in [2, 3]:

$$\phi_{\mathbf{C}}(s) = \phi_\tau(s) \cdot \phi_{\mathbf{X}}(s) \tag{22a}$$
$$\varphi_{\mathbf{C}}(s) = \varphi_\tau(s) + \varphi_{\mathbf{X}}(s) \tag{22b}$$
$$\kappa_r\{\mathbf{C}\} = \kappa_r\{\tau\} + \kappa_r\{\mathbf{X}\} . \tag{22c}$$

A matrix-variate Mellin convolution also exist (see [2,3]), but has been omitted here. These expressions are the foundation for application of the MKS framework to statistical analysis under the product model.

2.6. Method of log-cumulant estimation

The significance of the MKS framework lies in its ability to produce analytic expressions for pdfs, cumulative distribution functions, linear moments, logarithmic moments and mixed moments that would otherwise be difficult to derive. The appropriateness of performing statistical analysis with MKS manifests itself through the simplicity and the mathematical elegance of the expressions obtained. It gives an intuitive feeling that one is working in the right domain.

The most important practical implication of the theory is that low variance estimators for shape parameters of the product model distributions are easily derived from their log-cumulant equations (univariate case) and matrix log-cumulant equation (matrix-variate case). The well known method of moments (MoM) is based on using as many moment equations as there are unknown parameters, and to solve for the parameters by inverting the system of equations with sample moments inserted in place of population moments. Due to the particular moments applied, the implementation of this method under the MKS framework has been renamed the method of log-cumulants (MoLC) [1, 7]. Equivalently, it is called the method of matrix log-cumulants (MoMLC) in the matrix-variate case [2,3,8].

When there are more moment equations than unknown parameters, an optimisation problem can be formulated, which provides a solution known as the generalised method of moments (GMoM) [9]. The additional moments generally provide more information and have a positive effect on estimator bias and variance. In the GMoM, the covariance matrix of the sample moments is commonly used as a weighting matrix in the definition of the optimisation criterion. An example of such a procedure is the generalisation of the MoMLC described in [10].

Detailed descriptions of how the theory is turned into estimators for the paramaters of concrete distributions are found in the references. The estimators generally require numerical solution of equations that contain special functions, and polygamma functions in particular, but their implementation is otherwise straightforward.

3. THEORETICAL RESULTS

MKS have previously not been defined for SLC data, neither in the single polarisation nor the polarimetric case. We here provide an extension of the theory for polarimetric SLC data in the asymptotic case, that is, when the number of scattering vector samples is infinite.

3.1. Mellin kind statistics: single-look complex case

A salient property of the log-cumulants and matrix log-cumulants, as defined in the univariate and matrix-variate case, is that they are independent of scale from second order and onwards. This property is a direct result of utilising the logarithmic scale. In order to obtain the same property for logarithmic statistics of SLC data, the polarimetric whitening filter (PWF) has been chosen as the starting point of the derivation.

The PWF was proposed by Novak and Burl as a tool in both speckle filtering and target detection [11, 12]. It is defined as

$$y = \mathbf{s}^H \hat{\mathbf{\Sigma}}^{-1} \mathbf{s}\,, \tag{23}$$

where $\hat{\mathbf{\Sigma}}$ denotes an estimate of the covariance matrix $\mathbf{\Sigma} = E\{\mathbf{s}\mathbf{s}^H\}$. When invoking the product model from (6), this can be written as

$$\begin{aligned} y &= (\sqrt{\tau}\mathbf{x})^H \hat{\mathbf{\Sigma}}^{-1} (\sqrt{\tau}\mathbf{x}) \\ &= \tau \left(\mathbf{x}^H \hat{\mathbf{\Sigma}}^{-1} \mathbf{x}\right) = \tau\, q\,, \end{aligned} \tag{24}$$

by introducing the quadratic form q. With some background in multivariate statistics, the form of q will remind of a complex version of Hotelling's T^2 statistic, whose distribution is known [13]. For equality of q and Hotelling's T^2, $\mathbf{x}$ must be circular complex multivariate Gaussian, which is indeed the case. Moreover, it is required that $\hat{\mathbf{\Sigma}}$ is a complex Wishart matrix divided by its degrees of freedom. This is not as straightforward, and requires consideration when selecting the estimator for $\mathbf{\Sigma}$.

The PWF was originally implemented with the sample covariance matrix estimator

$$\hat{\mathbf{\Sigma}}_{\text{SCM}} = \frac{1}{n} \sum_{i=1}^{n} \mathbf{s}_i \mathbf{s}_i^H \tag{25}$$

given a sample $\mathcal{S} = \{\mathbf{s}_i\}_{i=1}^n$ of n independent and identically distributed (iid) scattering vectors. When $\mathbf{s} \sim \mathcal{N}_d^{\mathbb{C}}(\mathbf{0}, \mathbf{\Sigma})$, this estimator is known to be maximum likelihood (ML), unbiased and distributed as $\hat{\mathbf{\Sigma}}_{\text{SCM}} \sim \mathcal{W}_d^{\mathbb{C}}(n, \mathbf{\Sigma})$ [14]. In the general case, when $\mathbf{s}$ is non-Gaussian, then $\hat{\mathbf{\Sigma}}_{\text{SCM}}$ is neither ML nor Wishart distributed. A different estimator should therefore be applied.

The ML estimator for $\mathbf{\Sigma}$ under the general product model was first derived by Pulsone and Raghavan [15, 16]. The distribution of the ML estimate has not been determined. An asymptotic ML estimate was proposed independently by Gini et al. [17] and Conte et al. [18], which has become known as the fixed-point (FP) estimator for $\mathbf{\Sigma}$. Both the ML estimator and the FP estimator must be evaluated recursively, but while the former relies on model specific implementations with a relatively high computational cost, the FP estimator has a simple and model independent implementation given by

$$\hat{\mathbf{\Sigma}}_{\text{FP}} = \frac{d}{n} \sum_{i=1}^{n} \frac{\mathbf{s}_i \mathbf{s}_i^H}{\mathbf{s}_i^H \hat{\mathbf{\Sigma}}_{\text{FP}}^{-1} \mathbf{s}_i}\,. \tag{26}$$

Equally important, recent derivations by Pascal et al. have shown that the recursive implementation of (26) is globally convergent [19] and the asymptotic distribution of the FP estimate is [14]

$$\hat{\mathbf{\Sigma}}_{\text{FP}} \overset{n\to\infty}{\sim} \mathcal{W}_d^{\mathbb{C}}\left(\frac{d}{d+1}n, \mathbf{\Sigma}\right) \tag{27}$$

with the notation denoting convergence in distribution. Thus, the general asymptotic distribution of the FP estimator is the same as the finite sample distribution of the SCM estimator with Gaussian data, but it requires $(d+1)/d$ times as many samples.

Now that an estimator for $\mathbf{\Sigma}$ with the desired distribution has been found, let the FP-PWF be defined as

$$\begin{aligned} y &= \mathbf{s}^H \hat{\mathbf{\Sigma}}_{\text{FP}}^{-1} \mathbf{s} \\ &= \tau(\mathbf{x}^H \hat{\mathbf{\Sigma}}_{\text{FP}}^{-1} \mathbf{x}) = \tau\, Q \end{aligned} \tag{28}$$

by introducing the quadratic form Q. The goal of this derivation is to find the distribution of y and its MKS. Noting that y is decomposed as a product, where the distribution of τ is specified at will, it remains to determine the distribution of Q. Further assume that $\mathbf{x}$ and $\hat{\mathbf{\Sigma}}_{\mathrm{FP}}$ are statistically independent, then observe that Q can be decomposed into terms whose (asymptotic) distributions are

$$[\mathcal{N}_d^{\mathbb{C}}(\mathbf{0},\mathbf{\Sigma})]^H \times \left[\frac{\mathcal{W}_d^{\mathbb{C}}(n',\mathbf{\Sigma})}{n'}\right]^{-1} \times [\mathcal{N}_d^{\mathbb{C}}(\mathbf{0},\mathbf{\Sigma})] \quad (29)$$

with $n' = nd/(d+1)$. From Giri [13], the distribution of Q is thus known to be

$$\begin{aligned} Q &\overset{n\to\infty}{\sim} \frac{n'd}{n'-d+1} F_{2d,2(n'-d+1)} \\ &= \frac{nd}{n-d+(1/d)} F_{2d,2\left(\frac{d}{d+1}\right)(n-d+(1/d))} \end{aligned} \quad (30)$$

where $F_{a,b}$ denotes an F-distribution subscripted with its shape parameters a and b, or equivalently, its degrees of freedom.

The F-distribution is basically the same as the Fisher distribution or the Fisher-Snedecor distribution, although different parametrisations exist. Remark in particular that the Fisher distribution, as defined by Nicolas [7] and used repeatedly in the polarimetric SAR (PolSAR) literature in the same version (e.g., [5, 6, 20, 21]), has been extended with a location parameter m and uses shape parameters $\alpha = 2a$ and $\beta = 2b$. That is, the shape parameters are multiplied with a factor of two, when compared to the traditional definition of the F distribution (see e.g. [22, ch. 26]). The doubling of the shape parameters reflects that the traditional formulation of the F-distribution is rooted in real analysis, while the Fisher distribution used in the PolSAR literature is connected with intensities of complex variables. Thus, we can write

$$Q \overset{n\to\infty}{\sim} \frac{nd}{n-d+(1/d)} \mathcal{F}_{1,d,\left(\frac{d}{d+1}\right)(n-d+(1/d))} \quad (31)$$

by denoting the Fisher distribution (in Nicolas' version) as $\mathcal{F}(m,\alpha,\beta)$.

The log-cumulants of a Fisher variate were derived in [7], which yields

$$\begin{aligned} \kappa_1\{Q\} &= \psi^{(0)}(d) - \psi^{(0)}\left(\frac{d\left(n-d+\frac{1}{d}\right)}{d+1}\right) \\ &\quad + \log\left(\frac{nd}{(d+1)}\,\frac{(n-d-1)}{\left(n-d+\frac{1}{d}\right)}\right), \end{aligned} \quad (32a)$$

$$\begin{aligned} \kappa_{r>1}\{Q\} &= \psi^{(r-1)}(d) \\ &\quad - \psi^{(r-1)}\left(\frac{d\left(n-d+\frac{1}{d}\right)}{d+1}\right). \end{aligned} \quad (32b)$$

The log-cumulants of τ depend on $p_\tau(\tau)$, and have been derived for a number of different texture distributions in [2, 7]. This provides the underpinnings for the MoLC applied to SLC data.

3.2. Application to parameter estimation

By applying the relations of univariate MKS in (21) to the product model decomposition of the FP-PWF in (28), it is found that

$$\kappa_r\{y\} = \kappa_r\{\tau\} + \kappa_r\{Q\}\,. \quad (33)$$

The MoLC for SLC data consists of solving a set of log-cumulant equations defined by (33) for the unknown shape parameters in $p_\tau(\tau)$, denoted $\boldsymbol{\theta}$, with sample log-cumulants inserted for the population log-cumulants of y. This set of equations is defined by

$$\langle\kappa_r\{y\}\rangle - \kappa_r\{Q\} = \kappa_r\{\tau;\boldsymbol{\theta}\} \quad (34)$$

with $\langle\kappa_r\{y\}\rangle$ as the rth-order sample log-cumulant of y, and must contain as many equations (with different orders r) as there are unknown parameters.

A K-distributed scattering vector is written $\mathbf{s} \sim \mathcal{K}(\mathbf{\Sigma},\nu)$, showing that its distribution is parametrised by the covariance matrix $\mathbf{\Sigma}$ and the shape parameter ν, which is inherited from the unit mean and gamma distributed texture variable: $\tau \sim \gamma(1,\nu)$. The log-cumulants of τ are known to be [2, 7]

$$\kappa_1\{\tau\} = \psi^{(0)}(\nu) - \log(\nu)\,, \quad (35a)$$

$$\kappa_{r>1}\{\tau\} = \psi^{(r-1)}(\nu)\,. \quad (35b)$$

A practical algorithm for estimating ν is obtained by inserting (32) and (35) into (34) together with a sample log-cumulant obtained by combining sample log-moments computed from

$$\langle\mu_r\{y\}\rangle = \frac{1}{n}\sum_{i=1}^{n}(\log y)^r\,. \quad (36)$$

For instance, we may use the second-order log-cumulant equation, that is, (34) with $r = 2$. An interpretation of the procedure is given as follows: The second-order log-cumulant of y is the same as its variance computed on logarithmic scale. One then removes the logarithmic variance contributed by speckle, which amounts to $\kappa_2\{Q\}$. The remaining logarithmic variance is due to texture, depending solely on ν, which can be solved for by numerical inversion.

Some remarks should be made about MoLC estimation with SLC data. First, it appears from (34) that there are no nuisance parameters involved in the estimation of $\boldsymbol{\theta}$, even in the first-order log-cumulant equation. This is different from the multilook case, where the first-order log-cumulant contains the scale parameter, which severely limits its information content and value for estimation purposes (see [1–3]). In the single-look case, the scale parameter has effectively been removed by the whitening operation. This naturally nurtures a hope that the first-order equation is a suitable candidate for the MoLC procedure. However, $\hat{\mathbf{\Sigma}}_{\mathrm{FP}}$ only produces an estimate of the normalised $\mathbf{\Sigma}$. Thus, the scale parameter is still unknown and must be estimated separately [23], thereby reintroducing the nuisance parameter.

In the derivation of the FP estimator the following expression occurs, which is referred to as the ML estimate of τ given knowledge of the true $\mathbf{\Sigma}$:

$$\hat{\tau}_i = \frac{\mathbf{s}_i^H \mathbf{\Sigma}^{-1} \mathbf{s}_i}{d} \,. \tag{37}$$

The second remark goes to the feasibility of recovering the shape parameters of τ from a set $\mathcal{T} = \{\hat{\tau}_i\}_{i=1}^n$ computed from (37). In practice, the evaluation of (37) inevitably requires an estimate of $\mathbf{\Sigma}^{-1}$. Thus, the statistical variance within $\mathcal{T}$ is contributed by texture, as desired, but also by speckle through $\hat{\mathbf{\Sigma}}^{-1}$. Consequently, the variance of τ will be overestimated, and its shape parameter(s) underestimated. The appropriate alternative is described by (34), where the texture contribution is isolated by explicitly removing the speckle contribution, such that the correct shape parameter can be retrieved.

The third remark goes to the applicability of MoLC estimation based on the asymptotic MKS for SLC data. It is believed that the approach is well suited for estimation of clutter statistics. In target detection, the distribution of the background clutter is estimated with a sample size which is sufficiently large to defend the use of asymptotic statistics. It would be extremely attractive if the shape parameters could be estimated also within a multilooking cell, thereby providing an additional textural feature for image analysis. However, within a small neighbourhood the number of samples is low, which prompts the use of finite sample statistics. Moreover, there is considerable correlation between adjacent pixels, which further limits the approach. The extension of the SLC case MKS to finite and possibly correlated samples is considered future work.

4. EXPERIMENTAL RESULTS

This section presents preliminary results on MoLC estimation with SLC data. K-distributed data with shape parameter $\nu = 5$ is Monte Carlo simulated. The parameter is then estimated with two classical methods and the newly proposed MoLC estimator. Details on the estimators are given below, after noting that the linear moments of a scalar K-distributed single-look intensity Y are [1,7]

$$E\{Y^r\} = \left(\frac{\sigma}{\nu}\right)^r \frac{\Gamma(\nu + r)}{\Gamma(\nu)} \, r\,\Gamma(r) \tag{38}$$

where $\sigma = E\{Y\}$ and $\Gamma(\cdot)$ is Euler's gamma function.

MoM estimator: The MoM estimator is based on the expression $E\{Y^2\}/(E\{Y\})^2$. It can be solved for ν to yield the estimator

$$\hat{\nu}_{\text{MoM}} = \frac{2}{\langle Y^2 \rangle / \langle Y \rangle^2 - 2} \tag{39}$$

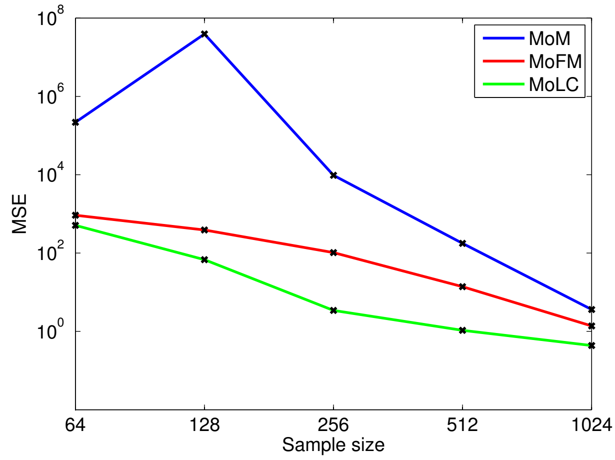

Figure 1. Mean squared error (MSE) of the MoM, MoFM and MoLC estimator produced in the estimation of the K-distribution shape parameter with true value $\nu = 5$.

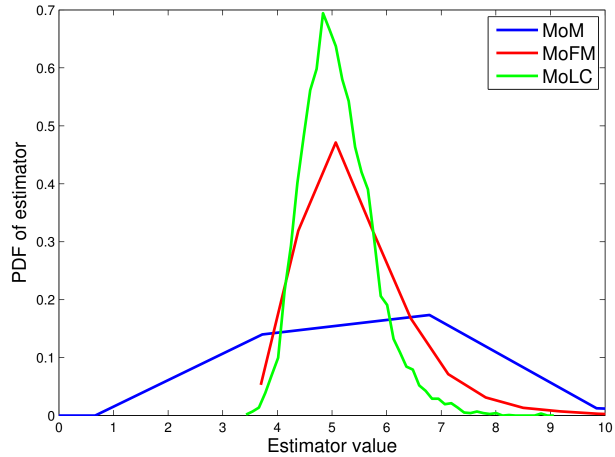

Figure 2. Estimated pdfs of the MoM, MoFM and MoLC estimator produced in estimation of the K-distribution shape parameter $\nu = 5$ with $n = 1024$ samples.

where $\langle Y^r \rangle$ is the rth-order sample moment of Y. The SLC data are first converted into single-look intensities and the estimate is computed for each polarimetric channel separately, before averaging over all channels.

MoFM estimator: The method of fractional moment (MoFM) estimator is based on the expression $E\{Y\}/(E\{Y^{1/2}\})^2$. The MoFM estimate of ν is obtained as the numerical solution of

$$\frac{\langle Y \rangle}{\langle Y^{1/2} \rangle^2} = \frac{4}{\pi} \frac{\hat{\nu}_{\text{MoFM}} \Gamma(\hat{\nu}_{\text{MoFM}})^2}{\Gamma(\hat{\nu}_{\text{MoFM}} + 1/2)^2} \,. \tag{40}$$

Also this estimator is applied to the single-look intensities in each channel separately, before averaging over the channel specific estimates.

MoLC estimator: As explained in Sections 2.6 and 3.2, different estimators can be constructed from (34). The second-order log-cumulant equation of y has been used here to obtain ν_{MoLC}, while being aware that improved estimators can be obtained by using more equations.

Figure 1 shows the mean squared error (MSE) of the estimators as a function of sample size. It clearly shows that the MoLC is much superior to the others for moderate to high sample sizes. It is also better for small samples sizes, but the lower performance relative to the other estimators can be explained by the inaccuracy of the asymptotic statistics, which becomes pronounced for small sample sizes. The results are obtained with 10,000 runs in the Monte Carlo simulations. Figure 2 displays the estimated PDF of the estimators for a sample size of $n = 1024$, which again demonstrates the superiority of the MoLC estimator.

5. CONCLUSIONS

The theory of MKS has been reviewed, emphasising the role of the logarithmic transformation and its connection with the Mellin transform in the statistical analysis of the doubly stochastic product model. The MKS framework has then been extended to cover polarimetric SLC data in the asymptotic case when the number of data points tends to infinity. The PWF implemented with the FP covariance matrix estimator is the key in this derivation. Application of the new theory to estimation of shape parameters in compound distributions for SLC data has been theoretically explained and practically demonstrated on simulated data. The simulation results show that the MoLC provides low variance estimators also for polarimetric SLC data.

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REFERENCES

1. J.-M. Nicolas, "Introduction aux statistique de deuxième espèce: Application des logs-moments et des logs-cumulants à l'analyse des lois d'images radar," *Traitement du Signal*, vol. 19, no. 3, pp. 139–167, 2002, in French, English translation in [2].

2. S. N. Anfinsen, "Statistical analysis of multilook polarimetric radar images with the Mellin transform," Ph.D. dissertation, University of Tromsø, Tromsø, Norway, May 2010.

3. S. N. Anfinsen and T. Eltoft, "Application of the matrix-variate Mellin transform to analysis of polarimetric radar images," *IEEE Trans. Geosci. Remote Sens.*, vol. 49, no. 7, Jul. 2011, 15 pp., in press.

4. A. M. Mathai, *Jacobians of Matrix Transformations and Functions of Matrix Arguments*. New York, USA: World Scientific, 1997, ch. 6.

5. G. Vasile, J.-P. Ovarlez, F. Pascal, and C. Tison, "Coherency matrix estimation of heterogeneous clutter in high-resolution polarimetric SAR images," *IEEE Trans. Geosci. Remote Sens.*, vol. 48, no. 4, pp. 1809–1826, Apr. 2010.

6. L. Bombrun, "Hierarchical segmentation of polarimetric SAR images using heterogeneous clutter models," *IEEE Trans. Geosci. Remote Sens.*, vol. 49, no. 2, pp. 726–737, Feb. 2011.

7. J.-M. Nicolas, "Application de la transformée de Mellin: Étude des lois statistiques de l'imagerie cohérente," Ecole Nationale Supérieure des Télécommunications, Paris, France, Tech. Rep. 2006D010, 2006, in French.

8. S. N. Anfinsen and T. Eltoft, "Analysis of multilook polarimetric radar data with the matrix-variate Mellin transform," in *Proc. EUSAR 2010 – 8th Eur. Conf. on Synthetic Aperture Radar*, Aachen, Germany, 7-10 Jun. 2010, 4 pp.

9. A. R. Hall, *Generalized Method of Moments*, ser. Advanced Texts in Econometrics. Oxford, U.K.: Oxford University Press, 2005.

10. S. N. Anfinsen, A. P. Doulgeris, and T. Eltoft, "Goodness-of-fit tests for multilook polarimetric radar data based on the Mellin transform," *IEEE Trans. Geosci. Remote Sens.*, vol. 49, no. 8, Aug. 2011, 18 pp., in press.

11. L. M. Novak and M. C. Burl, "Optimal speckle filtering in polarimetric SAR imagery," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 26, no. 2, pp. 293–305, Mar. 1993.

12. L. M. Novak, M. C. Burl, R. D. Chaney, and G. J. Owirka, "Optimal polarimetric processing for enhanced target detection," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 29, no. 1, pp. 234–244, Jan. 1993.

13. N. Giri, "On the complex analogues of T^2 and R^2 tests," *Ann. Math. Statist.*, vol. 36, no. 2, pp. 664–670, Apr. 1965.

14. F. Pascal, P. Forster, J.-P. Ovarlez, and P. Larzabal, "Performance analysis of covariance matrix estimates in impulsive noise," *IEEE Trans. Signal Process.*, vol. 56, no. 6, pp. 2206–2217, Jun. 2008.

15. R. S. Raghavan and N. B. Pulsone, "A generalization of the adaptive matched filter receiver for array detection in a class of a non-Gaussian interference," in *Proc. Adaptive Sensor Array Processing (ASAP) Workshop*, vol. 4. Lexington, U.S.: MIT Lincoln Laboratory, 13-15 March 2006, pp. 499–517.

16. N. B. Pulsone, “Adaptive signal detection in non-Gaussian interference,” Ph.D. dissertation, Northeastern University, Dept. of Electrical and Computer Engineering, Boston, U.S., May 1997.

17. F. Gini and M. Greco, “Covariance matrix estimation for CFAR detection in correlated heavy tailed clutter,” *Signal Processing*, vol. 82, no. 12, pp. 1847–1859, Dec. 2002.

18. E. Conte, A. De Maio, and G. Ricci, “Recursive estimation of the covariance matrix of a compound-Gaussian process and its application to adaptive CFAR detection,” *IEEE Trans. Signal Process.*, vol. 50, no. 8, pp. 1908–1915, Aug. 2002.

19. F. Pascal, Y. Chitour, J.-P. Ovarlez, P. Forster, and P. Larzabal, “Covariance structure maximum-likelihood estimates in compound gaussian noise: Existence and algorithm analysis,” *IEEE Trans. Signal Process.*, vol. 56, no. 1, pp. 34–48, Jan. 2008.

20. L. Bombrun and J.-M. Beaulieu, “Fisher distribution for texture modeling of polarimetric SAR data,” *IEEE Geosci. Remote Sens. Lett.*, vol. 5, no. 3, pp. 512–516, Jul. 2008.

21. O. Harant, L. Bombrun, G. Vasile, L. Ferro-Famil, and M. Gay, “Displacement tracking by maximum likelihood texture tracking,” *IEEE J. Sel. Topics Appl. Earth Observations Remote Sens.*, vol. 4, 2011, 12 p., in press.

22. N. L. Johnson and S. Kotz, *Continuous Univariate Distributions*. New York, U.S.: John Wiley & Sons, 1970, vol. 2.

23. G. Vasile, F. Pascal, and J.-P. Ovarlez, “Heterogeneous clutter model for high resolution polarimetric SAR data parameter estimation,” in *Proc. 5th Int. Workshop Sci. Appl. SAR Polarimetry and Polarimetric Interferometry (POLinSAR 2011)*, vol. ESA SP-695, Frascati, Italy, 24–28 Jan. 2011, in press.