ANALYSIS OF SAR IMAGES IN THE FRAMEWORK OF SCALE MIXTURE OF GAUSSIAN MODELS.

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ABSTRACT

In this paper we present the normal variance-mean mixture model as a framework for analyzing SAR data. The complex envelope of the echo signal is considered as a double stochastic circular Gaussian variable, in which both the variance and the mean are linearly scaled by a stochastic scaling factor Z. We then derive the generalized K amplitude model, and indicate how its parameters can be estimated from data. Some preliminary results show that this model represents the amplitude of SAR data well.

1. INTRODUCTION

Both *Gaussian* and *non-Gaussian* models have been considered for describing the statistics of SAR images. Non-Gaussian statistics will cause the amplitude distribution to deviate from the Rayleighian, and the most well-known non-Rayleigh amplitude model is the K distribution [1]. Among other non-Rayleigh models we mention the Nakagami distribution [2], the homodyned and the generalized K-distributions [3], and the more recent Rician Inverse Gaussian (RiIG) distribution [4, 5].

In this paper we introduce the so-called *normal variance-mean mixture models* [6] as distributions for representing the statistics of SLC SAR images. The normal variance-mean mixture models constitute a general approach for generating multi-dimentional non-Gaussian distributions, in which both the mean and the variance are linearly scaled by a stochastic scaling factor Z. We specifically study the case, in which the scaling variable is Γ-distributed, and show how the above mentioned K (standard, generalized) amplitude distributions actually are amplitude distributions associated with special cases of normal variance-mean mixture models.

The paper is organized as follows. In the next section, we present the normal variance-mean mixture models and derive the probability density function of the general multi-dimentional K distribution. In section 3, we discuss the generalized K model, and examines this model as an alternative to represent the local amplitude statistics of a SAR images. Finally in section 4, we give some conclusions.

2. NORMAL VARIANCE-MEAN MIXTURE MODELS

The normal variance-mean mixture models were introduced in [6]. A 1-D normal variance-mean mixture variable is in its most general form expressed as

$$Y = \mu + \beta Z + \sqrt{Z} X, \tag{1}$$

where X is a normalized Gaussian variable and Z is some positive random scaling variable. β is a scalar parameter. Hence, in this model, both the mean and the variance of Y are varying linearly as function of the stochastic variable Z.

A multidimensional extension is straight forward. Let $\mathbf{X}$ be a d-dimensional, zero mean Gaussian variable with covariance matrix equal to the identity matrix. Let furthermore, $\mathbf{\Gamma} \in \mathcal{R}^{d\times d}$ be a positive definite, symmetric, matrix with determinant $\det \mathbf{\Gamma} = 1$, and let Z be a scalar random variable with pdf $p_Z(z)$, which can attain only positive values. We now generate a new variable $\mathbf{Y}$ as *a multivariate variance-mean mixture variate* according to

$$\boldsymbol{Y} = \boldsymbol{\mu} + Z\mathbf{\Gamma}\boldsymbol{\beta} + \sqrt{Z}\mathbf{\Gamma}^{\frac{1}{2}}\boldsymbol{X}, \tag{2}$$

where $\boldsymbol{\mu}$ is a location vector, $\boldsymbol{\beta}$ is a vector parameter accounting for the linear scaling of the mean of $\boldsymbol{Y}$ as function of Z. The matrix $\mathbf{\Gamma}$ defines the internal covariance structure of the component variables of $\mathbf{Y}$. For this reason we will refer to this matrix as the covariance structure matrix. To obtain the marginal pdf of $\boldsymbol{Y}$, we have to perform an integration over the prior distribution $p_Z(z)$, which accordingly gives us

$$p_{\boldsymbol{Y}}(\boldsymbol{y}) = \int_0^\infty p_Z(z)\frac{1}{(2\pi\, z)^{\frac{d}{2}}} \times \exp\left(-\frac{(\boldsymbol{y}-\boldsymbol{\mu}-\mathbf{\Gamma}\boldsymbol{\beta}z)^t\mathbf{\Gamma}^{-1}(\boldsymbol{y}-\boldsymbol{\mu}-\mathbf{\Gamma}\boldsymbol{\beta}z)}{2z}\right)dz. \tag{3}$$

The superindex t denotes transpose of the matrix.

In general, probability density functions generated according to (2) will turn out as so-called sparse distributions, i.e., they are peaked at their mode, and have heavier tails than the Gaussian model.

2.1. The multivariate K distributions

In the sequel, we will examine the particular model which is associated with a Γ-distributed scale variable Z, i.e.

$$p_Z(z) = \left(\frac{\alpha}{\mu_Z}\right)^{\alpha} \frac{z^{\alpha-1}}{\Gamma(\alpha)} \exp\left(-\frac{\alpha z}{\mu_Z}\right) \tag{4}$$

Inserting this pdf into the integral of (3), we obtain a closed form expression for the pdf of $\boldsymbol{Y}$, which is explicitely given as

$$p_{\boldsymbol{Y}}(\boldsymbol{y}) = \frac{2}{\Gamma(\alpha)} \left(\frac{1}{2\pi}\right)^{\frac{d}{2}} \left(\frac{\alpha}{\mu_Z}\right)^{\alpha} \left(\frac{\delta}{\gamma}\right)^{\alpha-\frac{d}{2}} \times K_{\alpha-\frac{d}{2}}(\delta\gamma) \exp(\boldsymbol{y}-\boldsymbol{\mu})^t\boldsymbol{\beta} \tag{5}$$

where

$$\delta^2 = (\boldsymbol{y}-\boldsymbol{\mu})^t\boldsymbol{\Gamma}^{-1}(\boldsymbol{y}-\boldsymbol{\mu})$$

and

$$\gamma^2 = (\boldsymbol{\beta}^t\boldsymbol{\Gamma}\boldsymbol{\beta} + \frac{2\alpha}{\mu_Z}).$$

This model is a more general version of the multivariate K distribution that was proposed by Yueh et al. [7]. Using the facts that Z and $\boldsymbol{X}$ are statistical independent, the mean and covariance matrix of $\boldsymbol{X}$ are $\boldsymbol{\mu}_{\boldsymbol{X}} = \boldsymbol{0}$ and $cov\{\boldsymbol{X}\} = \boldsymbol{I}$ (the identity matrix), respectively, and that Z is Gamma distributed with pdf as in (4), we can easily obtain the mean and covariance matrix of $\boldsymbol{Y}$. These are given as

$$\boldsymbol{\mu}_{\boldsymbol{Y}} = \boldsymbol{\mu} + \mu_Z\boldsymbol{\Gamma}\boldsymbol{\beta}, \tag{6}$$

$$cov\{\boldsymbol{Y}\} = \mu_Z\boldsymbol{\Gamma} + \frac{\mu_Z^2}{\alpha}\boldsymbol{\Gamma}\boldsymbol{\beta}\boldsymbol{\beta}^t\boldsymbol{\Gamma}, \tag{7}$$

where we have also used that $\boldsymbol{\Gamma}$ is a symmetric matrix.

The moment generating function corresponding to (5) is given as:

$$M_{\boldsymbol{Y}}(\boldsymbol{\omega}) = E\{e^{\boldsymbol{\omega}^t\boldsymbol{Y}}\} = \frac{e^{\boldsymbol{\omega}^t\boldsymbol{\mu}}}{\left(1 - \frac{\mu_Z\boldsymbol{\omega}^t\boldsymbol{\Gamma}(\boldsymbol{\beta}+\frac{\boldsymbol{\omega})}{2}}{\alpha}\right)^{\alpha}}. \tag{8}$$

3. DERIVING THE GENERALIZED K DISTRIBUTION

In [3] the generalized and homodyned K-distributions were derived based on a biased random walk with fluctuating number of steps. The normal variance-mean mixture model presented in the previous section may be interpreted as a statistical formulation of a Brownian motion process with drift. In this section we show how the generalized K amplitude model can be derived as the amplitude distribution associated with a special 2-D K distribution. Let the complex backscattered signal Y be represented in terms of its quadrature components, i.e.

$$Y = Y_1 + jY_2 = Re^{j\Phi}, \tag{9}$$

where Φ is the phase, R is the amplitude, and Y_1 and Y_2 are referred to as the *in-phase* (I) and *quadrature* (Q) components, respectively. Let us consider Y as a 2-D normal variance-mean mixture signal, i.e. $\mathbf{Y} = [Y_1, Y_2]^t$, where $\mathbf{Y}$ is generated as in (2). We now consider X as a circular symmetric 2-D Gaussian variable with covariance matrix resempling the identity matrix, i.e. $X \sim N(0, I)$. We will furthermore assume that the covariance structure matrix $\boldsymbol{\Gamma}$ is given as the 2-D identity matrix, i.e.

$$\boldsymbol{\Gamma} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{10}$$

Note that, for a given value of the scale variable Z, i.e. Z is a deterministic constant, the Y_1- and Y_2-components are independent Gaussian variables with variances equal to z. However, when Z is a random scale variable, the unconditional distribution of Y_1 and Y_2 is non-Gaussian, and the variables are statistically dependent.

We now continue to derive the distribution for the amplitude R in the case when $\boldsymbol{\mu} = \mathbf{0}$ in (2), but $\boldsymbol{\beta}$ is non-zero. The covariance structure matrix is still the 2-D identity matrix, and the scale Z is Γ distributed as in (4).

Then we have

$$\mathbf{Y} = \boldsymbol{\beta}Z + \sqrt{Z}\mathbf{X}, \tag{11}$$

and the simultaneous pdf of $(Y_1, Y_2|Z)$ becomes

$$p_{Y_1,Y_2|Z}(y_1, y_2|Z) = \times \frac{1}{2\pi z} \exp\left(-\frac{(y_1-\beta_1 z)^2 + (y_2-\beta_2 z)^2}{2z}\right), \tag{12}$$

where $[\beta_1, \beta_2]^t = [\beta\cos(\omega), \beta\sin(\omega)]^t$. ω is the angle of $\boldsymbol{\beta}$ with respect to the Y_1-axis, and β is the norm of $\boldsymbol{\beta}$.

We now switch to polar coordinates. The amplitude is given as

$$R = \sqrt{Y_1^2 + Y_2^2}, \tag{13}$$

with the corresponding angle variable given as

$$\Phi = \tan^{-1}\left(\frac{Y_2}{Y_1}\right). \tag{14}$$

The resulting pdf for R, conditioned on Z, is obtained by integrating the simultaneous pdt of (R, Φ) with respect to Φ. This gives us

$$p_{R|Z}(r|Z) = \frac{r}{z} \exp\left(-\frac{r^2 + \beta^2 z^2}{2z}\right) I_0(\beta r), \tag{15}$$

where $I_0(\cdot)$ is the modified Bessel function of first kind. This is the well-known Rice distribution. The marginal distribution for R is obtained by integrating over the prior distribution for Z. Choosing Z to be Γ distributed, we get

$$p_R(r) = \left(\frac{\alpha}{\mu_Z}\right)^{\alpha} \frac{r}{\Gamma(\alpha)} I_0(\beta r) \times \int_0^{\infty} z^{\alpha-2} \exp\left(-\frac{1}{2}\left(\frac{r^2}{z} + (\beta^2 + \frac{2\alpha}{\mu_Z})z\right)\right) dz. \tag{16}$$

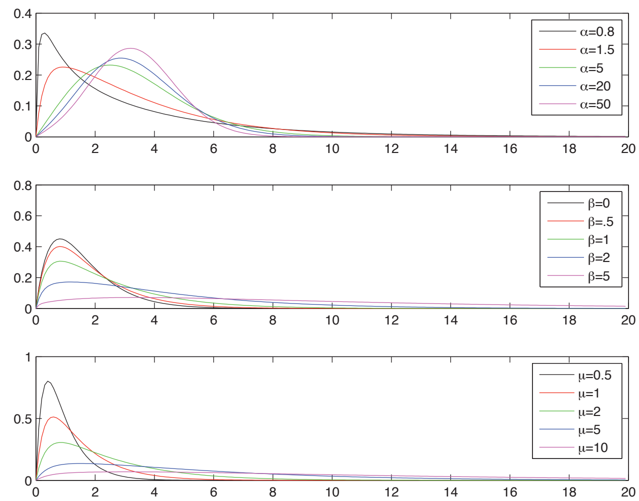

Fig. 1. Pdf-curves:Upper panel: several α values for $\beta = 1.5$ and $\mu = 2$; middle panel: several β values for $\alpha = 1.5$ and $\mu = 2$; lower panel: several μ values for $\alpha = 1.5$ and $\beta = 1$

This integral can be solved in closed form, and the resulting pdf is given as

$$p_R(r) = \frac{2}{\Gamma(\alpha)} \left(\frac{\alpha r}{\mu_Z}\right)^{\alpha} \frac{K_{\alpha-1}\left(r\sqrt{\beta^2 + \frac{2\alpha}{\mu_Z}}\right)}{\left(\sqrt{\beta^2 + \frac{2\alpha}{\mu_Z}}\right)^{\alpha-1}} I_0(\beta r), \tag{17}$$

where $K_\nu(\cdot)$ is the modified Bessel function of second kind, and order ν. The pdf in (17) is a normalized, valid probability density function, defined by the three parameters α, μ_Z, and β. This pdf model actually corresponds to the model, which is known in the literature as *the generalized K distribution*.

It is noted that the pdf in (17) has three parameters. This makes the pdf more fexible than the standard K distribution. β, the magnitude of $\boldsymbol{\beta}$, is a skewness parameter. The components of $\boldsymbol{\beta}$ hence account for any positive or negative skewness present in the marginal pdfs of the real and imaginary backscattered signal components. Physically β would be related to phase correlations that may exist between the contributions of scattering cells within the resolution area of the SAR. The α parameter is a measure of non-Gaussianity, and μ_Z, which is the mean of the random scaling factor Z, is related to the intensity of the signal. Fig.1 shows some curves, which illustrate how the shape of the pdf of the generalized K model varies with the parameters.

3.1. Parameter estimation

There are various alternatives for estimating the parameters of the generalized K distribution. The approach taken depends on which representation of the data is available. If we have available both quadrature components of the pixels, we can estimate the parameters of the full 2-D model from the data. This will in general give more reliable estimates. We include a present a method below, which is based on SLC data.

Note that the models discussed above is defined through a latent stochastic variable Z. The parameters can be estimated in an iterative maximum likelihood approach, using an EM type algorithm. In this case, the E-step involves updating the first and second order moments of $Z|\boldsymbol{y}$, and the M-step updates the parameters.
The latent variable Z is in our case Γ distributed, and using the expression in (4) for the pdf of Z, its k-th order moments are

$$E\{Z^k\} = \left(\frac{\mu_Z}{\alpha}\right)^k \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}. \tag{18}$$

From (18) we immediately see that μ_Z and α are obtained as

$$\mu_Z = E\{Z\}, \tag{19}$$

and

$$\alpha = \frac{1}{\frac{E\{Z^2\}}{E\{Z\}^2} - 1}. \tag{20}$$

Furthermore, using Bayes rule to (16), we find that the posterior distribution $Z|\boldsymbol{y}$ has a pdf given as

$$\begin{aligned} p_{Z|\boldsymbol{Y}}(z|\boldsymbol{Y}) &= \frac{p_{\boldsymbol{Y}|Z}(\boldsymbol{y}|Z)\,p_Z(z)}{p_{\boldsymbol{Y}}(\boldsymbol{y})} \\ &= \left(\frac{\sqrt{\beta^2 + \frac{2\alpha}{\mu_Z}}}{\sqrt{\boldsymbol{y}^t\boldsymbol{y}}}\right)^{\alpha-1} \frac{z^{(\alpha-1)-1}}{2K_{\alpha-1}(\sqrt{\boldsymbol{y}^t\boldsymbol{y}}\sqrt{\beta^2 + \frac{2\alpha}{\mu_Z}})} \\ &\quad \times \exp\left(-\frac{1}{2}(\frac{\boldsymbol{y}^t\boldsymbol{y}}{z} + (\beta^2 + \frac{2\alpha}{\mu_Z})z)\right). \end{aligned} \tag{21}$$

The expression for $p_{Z|\boldsymbol{Y}}(z|\boldsymbol{Y})$ is recognized as a Generalized Inverse Gaussian pdf with parameters $\{\alpha - 1, \sqrt{\boldsymbol{y}^t\boldsymbol{y}}, \sqrt{\beta^2 + \frac{2\alpha}{\mu_Z}}\}$, which we denote as $Z|\boldsymbol{Y} \sim \text{GIG}(z; \alpha - 1, \sqrt{\boldsymbol{y}^t\boldsymbol{y}}, \sqrt{\beta^2 + \frac{2\alpha}{\mu_Z}})$. The kth-order moments of a $\text{GIG}(u; \alpha, \delta, \gamma))$ distribution is [6]

$$E\{U^k\} = \left(\frac{\delta}{\gamma}\right)^k \frac{K_{\alpha+k}(\delta\gamma)}{K_\alpha(\delta\gamma)}. \tag{22}$$

From (11) we find that $E\{\boldsymbol{Y}\} = \boldsymbol{\beta}E\{Z\} = \boldsymbol{\beta}\mu_Z$.

Given an observation $\boldsymbol{y}_i$. Let $r_i = \sqrt{\boldsymbol{y}^t\boldsymbol{y}}$. From (22) we have that

$$\begin{aligned} \eta_i &= E\{Z|r_i\} \\ &= \frac{r_i}{\sqrt{\beta^2 + \frac{2\alpha}{\mu_Z}}} \frac{K_\alpha(r_i\sqrt{\beta^2 + \frac{2\alpha}{\mu_Z}})}{K_{\alpha-1}(r_i\sqrt{\beta^2 + \frac{2\alpha}{\mu_Z}})}, \end{aligned} \tag{23}$$

and

$$\xi_i = E\{Z^2|r_i\} = \left(\frac{r_i}{\sqrt{\beta^2 + \frac{2\alpha}{\mu_Z}}}\right)^2 \frac{K_{\alpha+1}(r_i\sqrt{\beta^2 + \frac{2\alpha}{\mu_Z}})}{K_{\alpha-1}(r_i\sqrt{\beta^2 + \frac{2\alpha}{\mu_Z}})}. \quad (24)$$

Given N observations, we define

$$\bar{\eta} = \frac{1}{N}\sum_{i=1}^{N}\eta_i, \quad (25)$$

and

$$\bar{\xi} = \frac{1}{N}\sum_{i=1}^{N}\xi_i. \quad (26)$$

Regarding $\bar{\eta}$ and $\bar{\xi}$ as estimates for $E\{Z\}$ and $E\{Z^2\}$, respectively, estimates for μ_Z and α may be obtained from (19) and (20).

Iterative algorithm

(i): Calculate $\hat{\boldsymbol{\mu}}_{\boldsymbol{Y}} = \frac{1}{N}\sum_{i=1}^{N}\boldsymbol{y}_i$.
(ii): Set $l = 0$. Select some initial estimates for the parameters $\hat{\alpha}_l$ and $\hat{\mu}_{Z,l}$.
(iii): Estimate $\hat{\boldsymbol{\beta}}_l$ as $\hat{\boldsymbol{\beta}}_l = \frac{\hat{\boldsymbol{\mu}}_{\boldsymbol{Y}}}{\hat{\mu}_{Z,l}}$. Set $l = l + 1$.
(iv): Calculate η_i and ξ_i using equations (23) and (24) with $r_i = \sqrt{y_i^t \boldsymbol{y}_i}$.
(v): Calculate $\bar{\eta}$ and $\bar{\xi}$ using (25) and (26).
(vi): Estimate $\hat{\mu}_{Z,l}$ and $\hat{\alpha}_l$ using (19) and (20).
(vii): Repeat steps *(iii)*- *(vii)* until convergence.

3.2. Results

We have done some preliminary tests to examine the appropriateness of the generalized K model to represent the amplitude statistics of the amplitude of SAR data. This was done by measuring the summed absolute value of the deviation between the model pdf and locally generated histograms. The standard K model was used as a reference. The data we used corresponds to the HH channel of the airborne L- band (1.25 GHz) PolSAR data set acquired with the EMISAR instrument over the agriculture test area of Foulum, Denmark, on April 17, 1998.model. It turned out that the generalized K model always ($> 99.99\%$) gave a better fit, even though in many cases the differences between the two models are insignificant. Fig. 2 shows an example of how the models fit a locale histogram.

4. CONCLUSION

We have presented the general framework of normal variance-mean mixture models for generating non-Gaussian statistical distributions. In the context of SAR scattering, this model corresponds to viewing the scattering process as a continuous Brownian motion with drift, as opposed to the usual discrete random walk model. In the paper, we have included a discussion of the generalized K amplitude model, and given an iterative procedure for estimating the parameters from data. The preliminary tests indicate that this model is superior to the standard K model in representing the amplitude statistics of SAR data.

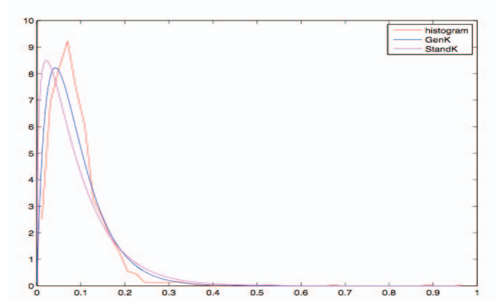

Fig. 2. An example of fit to a locale image amplitude histogram (red) of generalized K (blue) and standard K (magenta).

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